Algebras of Generalized Functions and Nonstandard Analysis

Hans Vernaeve
(joint work with Todor Todorov)

University of Innsbruck

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1. Generalized functions: introduction and motivation
   - Linear generalized functions (distributions)
   - Nonlinear generalized functions

2. Improving generalized functions by means of ultrafilters
   - Idea of construction
   - Properties
Linear generalized functions: Dirac’s $\delta$-impulse

- Physical interpretation: singular object with an infinite concentration at the origin $x = 0$, e.g. mass distribution of a unit point mass.
- Formal property: $\int_{\mathbb{R}^n} \delta(x) \varphi(x) \, dx = \varphi(0)$, for each $\varphi \in C^\infty(\mathbb{R}^n)$. (*)

Observation 1: The map $C^\infty_c(\mathbb{R}^n) \to \mathbb{R}: \varphi \mapsto \varphi(0)$ is a continuous linear map. This map captures the essence of the formal property (*).

Observation 2: For any (locally integrable) function $f$, the map $C^\infty_c(\mathbb{R}^n) \to \mathbb{R}: \varphi \mapsto \int_{\mathbb{R}^n} f(x) \varphi(x) \, dx$ is a continuous linear map. This map determines $f$ completely (up to measure zero).
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\( C^\infty_c(\mathbb{R}^n) = \{ \text{smooth functions with compact support} \} \)
Linear generalized functions: distributions

**Definition**

A continuous linear map $C_c^\infty(\mathbb{R}^n) \rightarrow \mathbb{R}$ is called a (Schwartz) distribution.

There exists a natural definition of partial differentiation on distributions, extending the classical definition for $C^1$-functions. Every distribution has partial derivatives $\partial_1, \ldots, \partial_n$ in this sense.
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**Applications**

- Justification of formulas containing derivatives of nondifferentiable functions used by physicists
- Theory of partial differential equations (PDEs): every linear PDE with constant coefficients has a distributional solution (L. Ehrenpreis, B. Malgrange, 1955).
- Formulation of Quantum Field Theory.
Multiplication of distributions

- Linear operations \((+, \partial_j, \int)\) can be defined naturally on distributions.
- Products and other nonlinear operations have no natural counterpart on the space of distributions.

Example: \(\delta^2, \sqrt{\delta}\) do not make sense as distributions.
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Yet:
- In theoretical physics, formal products of distributions are used (e.g., in quantum field theory, general relativity).
- Nonlinear PDEs with singular (discontinuous or distributional) data occur as models of real-world phenomena (e.g. in geophysics).

Need for a mathematical theory.
The algebra $\mathcal{G}$ of nonlinear generalized functions

**Idea**

- A (Colombeau) **nonlinear generalized function** $\in \mathcal{G}$ is constructed by means of a net (=family) of $C^\infty$-functions.
- $\mathcal{G}$ should contain the space of distributions.
- A product in $\mathcal{G}$ should be defined that coincides with the product of (sufficiently regular) usual functions.

$\mathcal{G}$ will be a differential algebra provided with an embedding (=injective morphism) of the space of distributions.
The algebra $\mathcal{G}$ of nonlinear generalized functions

Construction of $\mathcal{G}$ (J.F. Colombeau):

$\left(\mathcal{C}^\infty\right)^{(0,1)} := \{\text{nets of smooth functions indexed by a parameter } \varepsilon \in (0, 1)\}$.

To ensure an embedding of distributions with good properties, the nets are restricted by a growth condition:

$$\mathcal{A} = \{(u_{\varepsilon})_{\varepsilon} \in (\mathcal{C}^\infty)^{(0,1)} :$$

$$(\forall K \subset \subset \mathbb{R}^n)(\forall \alpha \in \mathbb{N}^n)(\exists N \in \mathbb{N})(\sup_{x \in K} |\partial^\alpha u_{\varepsilon}(x)| \leq \varepsilon^{-N}, \text{ for small } \varepsilon)\}.$$
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Two nets are identified if their difference belongs to the differential ideal

$$I = \{(u_\varepsilon)_\varepsilon \in A :$$
$$\left(\forall K \subset \subset \mathbb{R}^n\right)\left(\forall \alpha \in \mathbb{N}^n\right)\left(\forall m \in \mathbb{N}\right)\left(\sup_{x \in K} |\partial^\alpha u_\varepsilon(x)| \leq \varepsilon^m, \text{ for small } \varepsilon\right)\}.$$ 

By definition, $G = A / I.$
The algebra $\mathcal{G}$ of nonlinear generalized functions

Distributions are embedded into $\mathcal{G}$ by smoothing. The embedding preserves the vector space operations and $\partial_j$.

**Theorem (Nonlinear operations in $\mathcal{G}$)**

If $F \in C^\infty(\mathbb{R}^m)$ with all derivatives of polynomial growth and $u_1, \ldots, u_m \in \mathcal{G}$, the composition $F(u_1, \ldots, u_m) \in \mathcal{G}$ is well-defined and coincides with the usual composition if $u_1, \ldots, u_m \in C^\infty$.

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In particular, $\mathcal{G}$ solves the problem of **multiplication of distributions**.

The theorem is optimal, in the following sense:

**Theorem (Schwartz impossibility result)**

*One cannot construct a differential algebra $\mathcal{A}$ containing the distributions such that the product $u_1 \cdot u_2$ in $\mathcal{A}$ coincides with the usual product, if $u_1, u_2 \in C^k$ (for fixed $k \in \mathbb{N}$).*
The ring \( \tilde{\mathbb{R}} \) of generalized numbers

Let \( u \in \mathcal{G} \).

- \( \int_{\mathbb{R}^n} u(x) \, dx \) can be defined as a generalized number.
- The point value \( u(a) \) at \( a \in \mathbb{R}^n \) can be defined as a generalized number.
- The set of generalized numbers \( \tilde{\mathbb{R}} \) coincides with the set of generalized functions in \( \mathcal{G} \) with zero gradient.
- \( \tilde{\mathbb{R}} \) is a non-archimedean partially ordered ring that contains \( \mathbb{R} \).

Example: \( \delta(0) \in \tilde{\mathbb{R}} \), \( \int_{\mathbb{R}^n} \delta^2(x) \, dx \in \tilde{\mathbb{R}} \) are infinitely large numbers.
$\tilde{\mathbb{R}}$ is a partially ordered ring with zero divisors.

- Hard to interpret: the value of a generalized function can be a number not comparable with a real number?
- Hard to obtain results: e.g., the Hahn-Banach theorem, a basic tool in functional analysis, does not hold for Banach spaces over $\tilde{\mathbb{R}}$.

By means of ultrafilters, the algebraic properties of nonlinear generalized functions can be improved (M. Oberguggenberger, T. Todorov, 1998).
An improved version of $G$: idea of construction

Let $\mathcal{U}$ be a nontrivial ultrafilter on $(0, 1)$.
In the spirit of ultrafilter-models of nonstandard analysis, an algebra of generalized functions $G_\mathcal{U} := \mathcal{A}_\mathcal{U}/\mathcal{I}_\mathcal{U}$ can be defined, where

$$\mathcal{A}_\mathcal{U} = \{ (u_\varepsilon)_\varepsilon \in (C^\infty)^{(0,1)} :$$

$$(\forall K \subset \subset \mathbb{R}^n)(\forall \alpha \in \mathbb{N}^n)(\exists N \in \mathbb{N})(\sup_{x \in K} |\partial^\alpha u_\varepsilon(x)| \leq \varepsilon^{-N}, \ \mathcal{U}\text{-a.e.}) \},$$

$$\mathcal{I}_\mathcal{U} = \{ (u_\varepsilon)_\varepsilon \in \mathcal{A}_\mathcal{U} :$$

$$(\forall K \subset \subset \mathbb{R}^n)(\forall \alpha \in \mathbb{N}^n)(\forall m \in \mathbb{N})(\sup_{x \in K} |\partial^\alpha u_\varepsilon(x)| \leq \varepsilon^m, \ \mathcal{U}\text{-a.e.}) \}.$$

It can be checked that this modification does not destroy the desirable properties of $G$ (in particular, the good embedding of the distributions).
An improved version of $\mathcal{G}$: properties

Within $\mathcal{G}_U$:

- The generalized numbers are isomorphic with the nonstandard **field of asymptotic numbers** $\rho \mathbb{R}$ (A. Robinson, 1972).
- $\rho \mathbb{R}$ is a totally ordered, real closed field.
- $\mathcal{G}_U$ is isomorphic with an algebra of **pointwise**, infinitely differentiable functions $\rho \mathbb{R}^n \rightarrow \rho \mathbb{R}$.
- The **Hahn-Banach theorem** holds for Banach spaces over $\rho \mathbb{R}$.

Using principles from nonstandard analysis, problems can be solved more easily.
The full algebra $G_{\text{full}}$ of nonlinear generalized functions

Embedding of distributions in $G$

- Fix a particular net $(\varphi_\varepsilon)_\varepsilon$ that approximates $\delta$.
- The embedded image of a distribution $T$ is the net $(T \ast \varphi_\varepsilon)_\varepsilon$, approximating $T$.

The choice of the net $(\varphi_\varepsilon)_\varepsilon$ is not unique and represents one particular way to approximate $\delta$. If one is free to choose an approximation to solve a particular problem, $G$ can be used.
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### Embedding of distributions in $\mathcal{G}$

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If the solution of a problem needs to be independent of the approximation, the so-called **full algebra** $\mathcal{G}_{\text{full}}$ (J.-F. Colombeau, 1983) is used.

$u \in \mathcal{G}_{\text{full}}$ is a net of smooth functions **indexed by** $C^\infty_c(\mathbb{R}^n)$ (up to a certain identification).

### Embedding of distributions in $\mathcal{G}_{\text{full}}$ (canonical)

- The embedded image of a distribution $T$ is the net $(T \ast \varphi)_\varphi$. 
An improved version of $G_{\text{full}}$

- $G_{\text{full}} = A_{\text{full}} / I_{\text{full}}$, but $A_{\text{full}}, I_{\text{full}}$ do not lend themselves to an interpretation as sets of nets in which a certain growth property holds modulo a filter on $C_c^\infty(\mathbb{R}^n)$.
- Adapting the definition of $G_{\text{full}}$ to this requirement causes technical difficulties: it is no longer clear that the nets representing distributions $(T \ast \varphi)_\varphi \in A_{\text{full}}$!
- By a careful choice of an ultrafilter $\mathcal{U}$ on $C_c^\infty(\mathbb{R}^n)$, one can ensure that $(T \ast \varphi)_\varphi \in A_{\mathcal{U},\text{full}}$. The resulting algebra $G_{\mathcal{U},\text{full}}$ satisfies both the good algebraic properties of $G_{\mathcal{U}}$ and the good (canonical) embedding properties of $G_{\text{full}}$ (T. Todorov, H. Vernaeve, 2007\(^1\)).

To describe singular physical phenomena, generalized functions (distributions) were introduced.

When nonlinear operations are used, a more general theory of nonlinear generalized functions is needed.

Ultrafilters can be used to improve the algebraic properties of nonlinear generalized function algebras.

Reference for the theory of Colombeau nonlinear generalized functions: